Bosonization, vicinal surfaces, and hydrodynamic fluctuation theory

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Through a Euclidean path integral we establish that the density fluctuations of a Fermi fluid in one dimension are related to vicinal surfaces and to the stochastic dynamics of particles interacting through long range forces with inverse distance decay. In the surface picture one easily obtains the Haldane relation, and identifies the scaling exponents governing the low energy, Luttinger liquid behavior. For the stochastic particle model we develop a hydrodynamic fluctuation theory, through which in some cases the large distance Gaussian fluctuations are proved nonperturbatively. [S1063-651X(99)03212-2]

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I. INTRODUCTION

As pointed out by Haldane some time ago [1,2], spinless fermions in one dimension interacting through a short range potential have universal ground state correlations. The universal properties are computed on the basis of the Tomanaga-Luttinger Hamiltonian [3–5] where low energy characteristics turn out to be labeled by two free parameters, traditionally denoted as the renormalized sound velocity v_s and K. These parameters must be matched to the microscopic Fermi liquid under consideration. In fact, we will see that they are given by suitable second derivatives of the ground state energy per length. The two most prominent predictions of the Luttinger liquid scenario are the following.

(i) The momentum distribution behaves as

$$\langle a^\dagger(k)a(k)\rangle \simeq |k-k_F|^\alpha \operatorname{sgn}(k-k_F)+(\text{regular part}) \tag{1.1}$$

close to the Fermi momentum k_F. Compared to the noninteracting case, the Fermi fluid loses its gap at k_F, and the Fermi ''surface'' is retained as power law singularity only with anomalous exponent $\alpha=\frac{1}{2}[K+(1/K)-2]$.

(ii) The density fluctuations in the ground state are severely suppressed and strongly correlated. Nevertheless they have Gaussian statistics. This is a consequence of the bosonization of the density field, which is the basic observation leading to the exact solution of the Tomanaga-Luttinger Hamiltonian [6].

The anomalous momentum distribution (1.1) was studied by Benfatto *et al.* [7] through a rigorous implementation of a renormalization group zooming onto the Fermi surface. The present paper focuses on the density fluctuations (ii). We recall first that for the ideal Fermi fluid the structure function, i.e., the Fourier transform of the density-density correlations is given by

$$S(k)=\begin{cases}|k|/2\pi & \text{for } |k|\leqslant 2k_F\\ \rho & \text{for } |k|\geqslant 2k_F,\end{cases} \tag{1.2}$$

with $k_F=\pi\rho$ the Fermi momentum and ρ the density. The interaction smoothens the cusp at $2k_F$, similarly to Eq. (1.1), as proved recently by Benfatto and Mastropietro [8]. For density fluctuations the behavior near $k=0$ is of interest. Here the interaction has a much less spectacular effect. It only changes the openening angle of the cusp (and of course modifies the straight piece). Here we plan to go beyond the static two-point function and to study the small k,ω behavior of all n-point functions, including their frequency behavior, i.e., we plan to study the generating functional of the density field for large space-time distances, at which the Gaussian statistics should be recovered.

As will be explained in detail below, the density fluctuations are most conveniently investigated through the path integral for the world lines of the fermions. This form leads to two other physical interpretations. One may think of the world lines as steps of a vicinal surface, and use the statistical mechanics of surfaces. In this picture the Gaussian fluctuations are fairly immediate, and Haldane's parameters are identified as suitable second derivatives of the surface tension. In the second interpretation one regards the fermionic world lines as trajectories of particles whose motion is then governed by certain stochastic differential equations. Such stochastic particle systems are usually described through a hydrodynamic type fluctuation theory. In our case the forces between the particles decay like the inverse distance, and are therefore long ranged. We will develop a suitable modification of the standard hydrodynamic fluctuation theory. In the framework of a stochastic particle system, at least for some cases, we prove the Gaussian fluctuations without going through the perturbative double expansion in n and the interaction strength.

Different physical interpretations of the same theoretical model lead to alternative approximation schemes. Properties which look very deep in one formulation are physically obvious in another. We regard it as interesting that one-dimensional Fermi liquids allow for three distinct physical interpretations, and we try to explore their interconnections.

II. BASIC MODELS AND THEIR PATH INTEGRALS

Let us start with the two prototypical models.

(i) Fermions on a ring $[0,l]$ interacting through a short

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range potential. The Hamiltonian reads

$$H=-\sum_{j=1}^{N}\frac{1}{2}\frac{\partial^2}{\partial x_j^2}+\frac{1}{2}\sum_{i\neq j=1}^{N}V(x_i-x_j). \qquad (2.1)$$

We have set $m=1=\hbar$. V is a short range potential, and $x_j \in[0,l]$ with periodic boundary conditions. By the Pauli exclusion principle, the ground state wave function ψ has to vanish at $\{x_i=x_j\}$ (Dirichlet boundary conditions). For density fluctuations the sign changes in ψ do not matter. Therefore, equivalently, we may regard Eq. (2.1) as the Hamiltonian of bosons with a hard core. Formally, this corresponds to adding an infinitely strong repulsive δ potential to V.

(ii) Fermions on the periodic lattice $[1,\ldots,l]$. In second quantized form the Hamiltonian reads

$$H=\sum_{x=1}^{l}\{-a_x^\dagger a_{x+1}-a_{x+1}^\dagger a_x-\Delta a_x^\dagger a_x a_{x+1}^\dagger a_{x+1}\}, \qquad (2.2)$$

$a_{l+1}=a_1$. For future use, we stated only the particular case of a nearest neighbor interaction. Through the Jordan-Wigner transformation (2.2) turns into the XXZ Hamiltonian as

$$H=\sum_{x=1}^{l}\left\{-\frac{1}{2}(\sigma_x^1\sigma_{x+1}^1+\sigma_x^2\sigma_{x+1}^2)-\Delta\sigma_x^3\sigma_{x+1}^3\right\}, \qquad (2.3)$$

with periodic boundary conditions $\vec{\sigma}_{l+1}=\vec{\sigma}_1$, where $\vec{\sigma}_x$ are the Pauli spin matrices at site x.

Both Hamiltonians (2.1) and (2.3) generate a path integral for the statistical weight of fermionic world lines. For Eq. (2.1) the free measures P_o are N independent Brownian motions $x_1(t),\ldots,x_n(t)$. By the Pauli exclusion principle they are constrained not to cross, i.e.

$$x_j(t)<x_{j+1}(t) \mod l \quad \text{for all} \quad t,j=1,\ldots,N, \qquad (2.4)$$

which ensures the Dirichlet boundary condition at $\{x_i=x_j\}$. The statistical weight of the world lines is then given by

$$\frac{1}{Z}P_0\exp\left[-\frac{1}{2}\sum_{i\neq j=1}^{N}\int dt\, V(x_i(t)-x_j(t))\right]\chi_{NC}, \qquad (2.5)$$

where Z is the normalizing partition function and χ_{NC} is the indicator function of the set defined in Eq. (2.4). χ_{NC} restricts the path integral (2.5) to noncrossing paths only. In bosonic language this constraint corresponds to an infinitely repulsive δ interaction.

The statistical weight generated by Eq. (2.3) is the same as in Eq. (2.5). Only the Brownian motion $x_j(t)$ is to be replaced by a continuous time random walk on $[1,\ldots,l]$ with jump rate 1 to the right and left neighbors. In our example we picked the particular interaction potential $V(x)=-\Delta$ for $|x|=1$, and $V(x)=0$ otherwise.

One physical picture suggests itself immediately. We can think of the fermionic world lines as the trajectories of a stochastic particle system. In fact, in the limit $t\to\infty$ it will be a stationary Markov process. It is a diffusion process in case (i) and a jump process in case (ii). The quantum mechanical free Bose field of density fluctuations corresponds to space-time Gaussian fluctuations in a stochastic particle system. This is the usual hydrodynamic fluctuation theory as governed by a linear Langevin equation. Unfortunately, the noncrossing constraint results in repulsive long range $(1/x)$-type forces between the particles. Therefore the standard hydrodynamic theory, developed for short range forces, does not apply. In Sec. IV we will explain the required modifications, which are in fact surprisingly small.

An alternative physical picture is also well known [9,10], but less immediate. We think of $x_j(t)$ as a step of unit height for a vicinal surface. More explicitly, we introduce a height function $h(x,t)$ for the height of a crystal surface relative to the x,t reference plane. Then

$$\frac{\partial}{\partial x}h(x,t)=\sum_{j=1}^{N}\delta(x-x_j(t)). \qquad (2.6)$$

In the t direction the average slope vanishes, whereas in the x direction it is given by the particle density $\rho=N/l$. Periodic boundary conditions for the particles means that the surface is extended by repetition at average slope N/l along the x axis.

The stepped surface $h(x,t)$ fluctuates with a statistical weight given by Eq. (2.5). In general, surface fluctuations are expected to be governed by a free massless Gaussian field in the infrared limit with a strength determined by the matrix of second derivatives of the surface free energy. Since it coincides with the ground state energy of Hamiltonian (2.1), we have a direct way to identify the parameters in the Luttinger Hamiltonian. The Haldane relation [1] then follows as a simple consequence.

The surface picture also indicates the limitations of the Luttinger liquid concept. The surface free energy may have cusps and/or flat pieces. The former case corresponds to a roughening transition where logarithmic fluctuations are suppressed to order 1 fluctuations. The latter case is step bunching. The slope ''segregates'' into macroscopic regions. The steps are closer together than expected by naive counting.

To give a brief outline, in Sec. III we develop the surface picture in more detail. In Sec. IV we explore the hydrodynamic fluctuation theory for the dynamics of world lines and its consistency with the surface picture. We argue for the validity of a linear Langevin equation governing the density fluctuations. The approximations can be controlled for the Calogero-Sutherland model and for a general system with short range interaction provided the two-point function scales. The necessary computations are provided in the Appendixes.

III. SURFACE FLUCTUATIONS AND BOSONIZATION

For the sake of concreteness we first discuss fermions on a lattice, cf. Hamiltonians (2.2) and (2.3). The relations derived below are general, however. We use the σ^3 representation, and it is convenient to set up a corresponding notation. We define $\eta_x=(1+\sigma_x^3)/2$. Then $\eta_x=0$ and 1, and we interpret $\eta_x=1$ as a surface step at site x. A whole step configuration is denoted by η. Then H of Eq. (2.3) in the σ^3 representation becomes a linear operator acting on functions $f(\eta)$, and is given by

$$Hf(\eta)=-\sum_{x=1}^{l}(\eta_x-\eta_{x+1})^2 f(\eta^{xx+1})-\Delta\sum_{x=1}^{l}\eta_x\eta_{x+1}f(\eta), \tag{3.1}$$

where the periodic boundary condition $\eta_{l+1}=\eta_1$ is understood. η^{xx+1} stands for the configuration η with occupations at sites x and $x+1$ interchanged. The path integral is generated by the transfer matrix $(e^{-tH})_{\eta\eta'}$, $t\geqslant 0$. For an isolated step the first term in Eq. (3.1) yields a symmetric, time continuous random walk with a jump rate 1 to its neighbors. For several steps $(\eta_x-\eta_{x+1})^2$ ensures the noncrossing constraint. $-\Delta\Sigma_{x=1}^{l}\eta_x\eta_{x+1}$ is a potential. Thus we see that Eq. (3.1) indeed generates a path integral of the form of Eq. (2.5) with $x_j(t)$ replaced by $\eta_x(t)$ and P_0 standing for independent random walks on the lattice $[1,\ldots l]$. For later use we define the steps in space-time by $\eta_x(t)=1(0)$, if at ''time'' t there is a (no) step at site x, $x=1,\ldots,l$ and $0\leqslant t\leqslant T$. The steps have a statistical weight given by Eq. (2.5). Clearly the number of steps is $N=\Sigma_{x=1}^{l}\eta_x(t)$ independent of t. In the dynamic picture we regard $\eta_x=1$ as a particle and $\eta_x=0$ as no particle at site x . The surface steps are then the world lines of the particles. Particles jump to their neighboring sites, but are never created and destroyed. We will use ''step'' and ''particle'' interchangeably.

In the crystallographic literature our model is known as terrace-step-kink model. It describes a high symmetry crystal surface miscut by a small angle. Such a vicinal surface consists of a regular array of monoatomic steps. Through thermal activation the steps meander but they do not cross or terminate. The slope of the vicinal surface imposes the step density and their average orientation. The terraces are the constant height pieces between steps and kink refers to a step corner.

The surface defined through Eq. (2.6), with $x_j(t)$ replaced by $\eta_x(t)$, is tilted in the x direction with slope $\rho=N/l$. A complete picture emerges only if we tilt the surface also along the t direction. To do so, we define the step current $J_{xx+1}(t)$ through the bond $(x,x+1)$. $J_{xx+1}(t)$ is a sequence of δ functions located at those times when a step jumps between x and $x+1$. The δ function carries a weight $+1$ (-1) if the jump is from x to $x+1$ $(x+1$ to $x)$. The tilt along the t coordinate is enforced by the additional constraint

$$\sum_{x=1}^{l}\int_0^T J_{xx+1}(t)dt=N\alpha T, \tag{3.2}$$

which implies that on the average each step has the drift α.

If the step variables $\eta_x(t), x=1,\ldots,l,\quad 0\leqslant t\leqslant T$, are given, then, by definition

$$h(x+1,t)-h(x,t)=\eta_x(t). \tag{3.3}$$

Equation (3.3) can be integrated to yield

$$h(x,t)=\sum_{y=1}^{x}\eta_y(0)-\int_0^t J_{xx+1}(s)ds, \tag{3.4}$$

with an arbitrary choice for the constant of integration.

At this point it is convenient to go from the canonical prescription to the grand canonical prescription. The variable conjugate to the number of particles is the chemical potential μ, and we introduce λ as conjugate variable to the total current (3.2). Thereby we obtain the transfer matrix $(e^{-tH})_{\eta\eta'}$, with the Hamiltonian

$$\begin{aligned}Hf(\eta)=&-\sum_{x=1}^{l}\{(e^{\lambda}\eta_x(1-\eta_{x+1})+e^{-\lambda}(1-\eta_x)\eta_{x+1})\\&\times f(\eta^{xx+1})+\Delta\eta_x\eta_{x+1}f(\eta)-\mu\eta_x f(\eta)\}\\=&H_l(\mu,\lambda)f(\eta).\end{aligned} \tag{3.5}$$

The exponential of the current gives weight e^{λ} to a right jump and weight $e^{-\lambda}$ to a left jump and the exponential of the particle number yields the potential $\mu\Sigma_x\eta_x$. Equation (3.5) is the Hamiltonian of the asymmetric XXZ model.

To make H symmetric we have to continue λ analytically to the imaginary axis, i.e. to replace λ by $-i\lambda$ with λ real. Tracing back to the Fermi Hamiltonian (2.2), we obtain

$$\begin{aligned}H=\sum_{x=1}^{l}\{&-e^{i\lambda}a_x^{\dagger}a_{x+1}-e^{-i\lambda}a_{x+1}^{\dagger}a_x\\&-\Delta a_x^{\dagger}a_x a_{x+1}^{\dagger}a_{x+1}+\mu a_x^{\dagger}a_x\}.\end{aligned} \tag{3.6}$$

This means that the dispersion relation $-2\cos k$ is replaced by $-2\cos(k-\lambda)$. In the low energy limit the dominant contributions have a total momentum $\langle N\rangle\lambda$. Thus the (analytically continued) λ regulates the average fermionic current on the ring $[1,\ldots,l]$, $\lambda=0$ corresponding to zero current.

The thermodynamics of the surface is governed by the surface tension $\sigma(\mathbf{u})$ depending on the slope $\mathbf{u}=(\rho,\rho\alpha)$. σ is convex up. To relate this to Hamiltonian (3.5), it is convenient to define the Legendre transform $\hat{\sigma}$ of σ by

$$\hat{\sigma}(\mathbf{v})=\inf_{\mathbf{u}}(\sigma(\mathbf{u})-\mathbf{v}\cdot\mathbf{u}). \tag{3.7}$$

$\hat{\sigma}$ is convex down, and in terms of H in Eq. (3.5) it is defined by

$$\hat{\sigma}(\mu,\lambda)=-\lim_{T,l\to\infty}\frac{1}{lT}\operatorname{tr}\exp[-TH_l(\mu,\lambda)]. \tag{3.8}$$

If we take first the limit $T\to\infty$, then Eq. (3.8) becomes $E_l(\mu,\lambda)/l$, with E_l the ground state energy of H_l. Thus the surface tension is simply the Legendre transform of the ground state energy of H_l per site:

$$\hat{\sigma}(\mu,\lambda)=\lim_{l\to\infty}\frac{1}{l}E_l(\mu,\lambda). \tag{3.9}$$

Thermodynamic fluctuation theory suggests that a small height fluctuation δh relative to the average slope $\mathbf{u}$ has a probability proportional to

$$\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}\sigma_{ij}(\mathbf{u})(\nabla_i\delta h)(\nabla_j\delta h)\right], \tag{3.10}$$

where $\sigma_{ij}(\mathbf{u})=\partial^2\sigma(\mathbf{u})/\partial u_i\partial u_j$. Thus on a large scale the height fluctuations should be Gaussian with a covariance

$$\left(\sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{u}) k_i k_j\right)^{-1} \tag{3.11}$$

in Fourier space. Equation (3.11) is the covariance of a free massless bosonic field in its Euclidean version. Fluctuations of two-dimensional surfaces grow logarithmically and are therefore not stationary. They become stationary by taking an x derivate, which according to Eq. (3.3) yields the step density. In Fourier space we only have to multiply by k_1^2. The thermodynamic fluctuation theory for surfaces predicts the low energy behavior of the density fluctuations for the world lines of Eq. (3.5).

We find it convenient to keep the t dependence and use k for the Fourier transform with respect to x. Then, according to Eqs. (3.11) and (3.3), we have

$$\langle \eta_x(t)\eta_{x'}(0)\rangle - \rho^2 \simeq \int dk\, e^{ik(x-x')}\tfrac{1}{2}c|k| e^{-\gamma|k||t| - ikvt} \tag{3.12}$$

for large $|x-x'|$ and $|t|$, where the parameters are defined by

$$\sigma_{11} = (\gamma^2+v^2)/\gamma c, \quad \sigma_{12} = -v/\gamma c, \quad \sigma_{22} = 1/\gamma c. \tag{3.13}$$

The density fluctuations are Gaussian on the scale where Eq. (3.12) is valid.

For the expression in Eq. (3.10) to make sense, $\sigma(\mathbf{u})$ must be twice differentiable at $\mathbf{u}$ and the matrix of second derivatives $D^2\sigma(\mathbf{u}) > 0$. Already for the simple model (3.5) with nearest neighbor step-step interaction only, this condition is not always satisfied. σ is known from the Bethe ansatz [11]. For $\Delta > -2$ at $\rho = \frac{1}{2}$, $\alpha = 0$, the steps align in antiferromagnetic order. Thereby surface fluctuations are strongly suppressed, from logarithmic order to order 1. Changing either ρ or α destroys this roughening transition. We refer to Ref. [12] for the behavior close to the transition.

On the attractive side, $\Delta > 0$, steps may bunch to give macroscopic patches of slope $\mathbf{u} = (1,0)$ and slope $\mathbf{u} = (0,0)$. This phase is bordered by the stochastic line, where $-H$ of Eq. (3.5) generates a stochastic time evolution [11]. Clearly, the condition is

$$\Delta = 2\cosh\lambda \tag{3.14}$$

for all ρ. To reexpress Eq. (3.14) in terms of α, we need $\alpha = \partial e/\partial\lambda$, which is not known in closed form. For small λ from linear response, we have $\alpha = \rho(1-\rho)2\sinh\lambda$, which implies that, for small α,

$$\Delta_c = 2 + (\alpha/2\rho(1-\rho))^2. \tag{3.15}$$

Coming back to Eq. (3.12), we note that the density fluctuations decay as $\gamma|k|$ and propagate with velocity v. The static $(t=0)$ covariance is $\frac{1}{2}c|k|$, which reflects that steps are stiffly arranged because of the noncrossing constraint. The parameters γ, v, and c can be expressed through the ground state energy as

$$\begin{aligned} c^2 &= \det D^2\hat{\sigma} = (\partial^2 e/\partial\mu^2)(\partial^2 e/\partial\lambda^2) - (\partial^2 e/\partial\mu\partial\lambda)^2, \\ \gamma &= c/(\partial^2 e/\partial\mu^2), \quad v = (\partial^2 e/\partial\mu\partial\lambda)/(\partial^2 e/\partial\mu^2), \end{aligned} \tag{3.16}$$

where we used Eqs. (3.9) and (3.13). These relations are valid in general.

Haldane [1] observed that for Luttinger fluids the parameters in the low energy effective bosonic action are not independent. This is easily rederived from Eq. (3.16). In the notation of Haldane,

$$v_s = \gamma, \quad v_N = \delta\mu/\delta\rho. \tag{3.17}$$

The average current is $j = \partial e/\partial\lambda$ and in linear response $j(\lambda) = j(\lambda_0) + v_J(\lambda_0)(\lambda - \lambda_0)$, i.e.,

$$v_J = \partial^2 e/\partial\lambda^2. \tag{3.18}$$

The Haldane relation reads

$$v_N v_J = v_s^2, \tag{3.19}$$

i.e.,

$$(\partial^2 e/\partial\mu^2)^{-1}(\partial^2 e/\partial\lambda^2) = \gamma^2, \tag{3.20}$$

which is to be compared with

$$(\partial^2 e/\partial\mu^2)^{-1}(\partial^2 e/\partial\lambda^2) = \gamma^2 + v^2 \tag{3.21}$$

by Eq. (3.16). Thus the Haldane relation holds at $v=0$, equivalently at $\lambda = 0$, which he used implicitly by setting $j(\lambda_0) = 0$ in the linear response. The parameter K mentioned in Sec. I is given by

$$K = \sqrt{v_J/v_N} = \left(\frac{\partial^2 e}{\partial\lambda^2}\frac{\partial^2 e}{\partial\mu^2}\right)^{-1/2}. \tag{3.22}$$

For $\lambda = 0$, Hamiltonian (3.5) is symmetric. In the dynamic picture we have a time-reversible jump process for the particles for which detailed balance is satisfied. As in the short range case this gives rise to an Einstein relation, and the Haldane relation can be viewed as a particular case.

For Brownian steps, as governed by the Hamiltonian (2.1) with Dirichlet conditions at $\{x_i = x_j\}$, some simplifications compared to the general case occur. The drift (equal to the tilt along t) is enforced by the constraint

$$\int_0^T dt\, \dot{x}_j(t) = \alpha T, \tag{3.23}$$

$j = 1,\ldots,N$. Such a drift can be trivially removed by the global change of coordinates $y_j(t) = x_j(t) - \alpha t$. In contrast to the terrace-step-kink model, the tilting costs only in elastic energy for each step individually. Let E_N be the ground state energy of Eq. (2.1), and define

$$e(\rho) = \lim_{N,l\to\infty, N/l=\rho} \frac{1}{l}E_N. \tag{3.24}$$

Then the ground state energy per unit volume at drift α is given by

$$e(\rho,\alpha)=e(\rho)+\tfrac{1}{2}\rho\alpha^2. \tag{3.25}$$

Comparing with Eq. (3.13) and using $\mathbf{u}=(\rho,\alpha\rho)$, we obtain $v=\alpha$, as expected, and

$$c=\sqrt{\rho/e''(\rho)},\qquad \gamma=\sqrt{\rho e''(\rho)}. \tag{3.26}$$

The Haldane relation is $c\gamma=\rho$.

We conclude this section by stating a precise conjecture. For the terrace-step-kink model we average over ε^{-1} sites with some smooth test function *f*. From the surface picture, the *x* and *t* coordinates must be on equal footing. Hence time is also scaled as ε^{-1}, and we introduce the fluctuation field

$$\xi^{\varepsilon}(f,t)=\sum_{x} f(\varepsilon x)(\eta_x(\varepsilon^{-1}t)-\rho). \tag{3.27}$$

Note that we are not in the standard central limit theorem setting. We sum over ε^{-1} sites, but expect a fluctuation of order 1 only. In the same spirit, for the Brownian steps we define the fluctuation field as

$$\xi^{\varepsilon}(f,t)=\sum_{j} f(\varepsilon x_j(\varepsilon^{-1}t))-\rho\int dx\, f(\varepsilon x). \tag{3.28}$$

Conjecture. We consider the path measure generated by Hamiltonian (3.1) in the limit $T\to\infty$ at fixed tilt α, cf. Eq. (3.2), and in the limit $l\to\infty$ at a fixed density $\rho=N/l$. This path measure governs the fluctuation field (3.27). If $e(\rho,\alpha)$ or $e(\mu,\lambda)$, respectively, is sufficiently smooth (at least twice differentiable), then

$$\lim_{\varepsilon\to 0}\ \xi^{\varepsilon}(f,t)=\xi(f,t) \tag{3.29}$$

in distribution. The limit is jointly Gaussian with covariance

$$\langle \xi(f,t)\xi(g,0)\rangle=\int dk\frac{1}{2}c|k|e^{-\gamma|k||t|}e^{-ikvt}\hat{f}(k)^{*}\hat{g}(k). \tag{3.30}$$

The parameters c,γ, and v are given by Eq. (3.16). Correspondingly, for fermions in the continuum with Hamiltonian (2.1) and path measure (2.5), the fluctuation field (3.28) satisfies limit (3.29) with $v=\alpha$ in Eq. (3.30).

IV. HYDRODYNAMIC FLUCTUATION THEORY

In the limit $T\to\infty$, for fixed *l*, the statistics of the steps become stationary in *t* and are governed by a stochastic process which by construction is Markov. To determine its generator, let ψ be the ground state and *E* the ground state energy for *H* of either Eqs. (2.1) or (3.1), which satisfy

$$H\psi=E\psi. \tag{4.1}$$

In the case of Eq. (2.1), we have $\psi>0$ except at coinciding points $x_j=x_i$, $i\neq j$, where $\psi(x)=0$. For Eq. (3.1) we first have to specify the sector $\Sigma_{x=1}^{l}\eta_x^3=N$. Then, for fixed *N*, e^{-tH} has a strictly positive integral kernel, and by the Perron-Frobenius theorem the ground state ψ is unique and $\psi>0$. The (backwards) generator of the Markov process for the steps is defined by

$$Lf=-\psi^{-1}(H-E)\psi f \tag{4.2}$$

as acting on functions *f* over the configuration space. The Markov process has ψ^2 as a unique invariant measure.

We carry out this construction for the *XXZ* model in the σ^3 representation and obtain, in the notation of Eq. (3.1),

$$Lf(\eta)=\sum_{x=1}^{l}(e^{\lambda}\eta_x(1-\eta_{x+1})+e^{-\lambda}(1-\eta_x)\eta_{x+1})c_{xx+1}(\eta)\times[f(\eta^{xx+1})-f(\eta)]. \tag{4.3}$$

L is the generator for a stochastic lattice gas, where particles jump to their neighboring sites. The exchange rate $c_{xx+1}(\eta)$ between sites *x*and $x+1$ is given by

$$c_{xx+1}(\eta)=\psi(\eta^{xx+1})/\psi(\eta). \tag{4.4}$$

The jumps to the right are biased by the factor e^{λ}, and those to the left by $e^{-\lambda}$. If $\lambda=0$, *H* is symmetric, and the stochastic evolution satisfies detailed balance.

Rates (4.4) are determined through the ground state, which is not known in general. Only for $\Delta=0$ does one have an explicit ground state ψ. If we denote by $x_1,\ldots,x_N$ the positions of the particles in the sector $\Sigma_{x=1}^{l}\eta_x=N$, then, in this sector,

$$\psi(x_1,\ldots,x_N)=\prod_{i<j=1}^{N}|\sin(\pi(x_i-x_j)/l)|. \tag{4.5}$$

Therefore the exchange rates are given by

$$c_{xx+1}^{(N)}=\exp\left[-(\eta_{x+1}-\eta_x)\sum_{\substack{y=1\\ y\neq x,x+1}}^{l}\times\left(\ln\left(\frac{\sin(\pi(y-x+1)/l)}{\sin(\pi(y-x)/l)}\right)\right)\eta_y\right]. \tag{4.6}$$

Taking formally the infinite volume limit yields

$$c_{xx+1}=\exp\left[-(\eta_{x+1}-\eta_x)\sum_{y\neq x,x+1}\left(\ln\left(\frac{y-x+1}{y-x}\right)\right)\eta_y\right]. \tag{4.7}$$

Equation (4.7) teaches us several points. The rates are repulsive: more particles to the left of *x* favors a right jump of the particle at *x*. The rates are long ranged, which means that the finite range intuition is no longer applicable. Since $\ln((y-x-1)/y-x)\cong-(y-x)^{-1}$, at infinite volume the rates may be infinite. The dynamics with rates (4.7) is then defined only for $|\psi|^2$ for almost all configurations. Because the case $\Delta=0$ maps to a free fermion theory, one can construct the infinite volume ground state as a measure on particle configurations and the Markov semigroup e^{Lt} in the corresponding L^2 space [13,14]. This implies an almost certain existence of the dynamics. If one adds the nearest neighbor interaction Δ, one expects exchange rates to be qualitatively similar to Eq. (4.7). We are not aware of any result in this direction.

For the Brownian steps L is the generator of a diffusion process given by

$$Lf(x)=\sum_{j=1}^{N}\left(a_j(x)\frac{\partial}{\partial x_j}+\frac{\partial^2}{\partial x_j^2}\right)f(x), \tag{4.8}$$

$x=(x_1,\ldots,x_N)$. The drift on the jth particle is

$$a_j=-\frac{\partial}{\partial x_j}\ln\psi. \tag{4.9}$$

Thus $\ln\psi$ is the potential for the diffusion process.

For Brownian steps with exclusion only, i.e., $V=0$, one finds that

$$a_j(x)=\sum_{i=1,i\neq j}^{N}\cot(\pi(x_j-x_i)/l). \tag{4.10}$$

For $l\to\infty$ this expression converges to

$$a_j(x)=\sum_{i,i\neq j}\frac{1}{x_j-x_i}, \tag{4.11}$$

i.e. to a repulsive $1/x$ force as for the lattice gas. The model with drift (4.11) was introduced by Dyson [15]. Its potential is formally given by

$$-\sum_{i\neq j}\ln|x_i-x_j|. \tag{4.12}$$

In fact, Dyson added a confining external potential, which is quadratic, and considered Eq. (4.11) for finite N.

We conclude that density fluctuations in the fermionic ground state may equally well be studied through the stochastic dynamics (4.3) and (4.8). If the potential is short ranged, there is a well developed machinery known as hydrodynamic fluctuation theory. One argues (and proves in many model systems [16,17]) that density fluctuations relax diffusively and are driven by a white noise random flux. This corresponds to the linear Langevin equation

$$\frac{\partial}{\partial t}\xi(x,t)=D\frac{\partial^2}{\partial x^2}\xi(x,t)+\sqrt{\sigma}\,\frac{\partial}{\partial x}W(x,t). \tag{4.13}$$

D is the bulk diffusion coefficient, and σ is the bulk mobility. They are related through $\sigma=\chi D$ with χ the static compressibility of the lattice gas. $W(x,t)$ is space-time white noise. The spatial derivative ensures the conservation of the number of particles. The stationary measure for Eq. (4.13) is Gaussian with a covariance matrix $(\sigma/2D)$. In passing we mention that in the short range case (4.13) holds only for $\lambda=0$, i.e. in the symmetric case. For $\lambda\neq 0$ one switches over to the Kardar-Parisi-Zhang (KPZ) universality class. It is characterized by the dynamic exponent $z=3/2$, rather than $z=2$ as in Eq. (4.13). The fluctuations are non-Gaussian [19].

Returning to the long range case of interest here, we have already obtained the limiting Gaussian fluctuations in the conjecture. We write $\xi(f,t)=\int dx\, f(x)\xi(x,t)$, and note that Eq. (3.30) determines a semigroup in t. Therefore, Eq. (3.30) must be the (stationary) solution of the linear Langevin equation

$$\frac{\partial}{\partial t}\xi(x,t)=\left(-\gamma\sqrt{-\partial^2/\partial x^2}-v\frac{\partial}{\partial x}\right)\xi(x,t)+\sqrt{c\gamma}\,\frac{\partial}{\partial x}W(x,t), \tag{4.14}$$

which is a surprisingly minimal modification compared to the short range case [Eq. (4.13)]. γ plays the role of D, and c that of σ. The crucial difference is that a Fourier mode e^{ikx} decays as $e^{-\gamma|k||t|}$ rather than as $e^{-Dk^2|t|}$.

V. SCALING LIMIT

We support our general conjecture by arguing that the stochastic particle evolution is close to the Langevin equation (4.14). We will do so on a fairly formal level. In particular, we simply work in infinite volume. In the Appendixes we explain how parts of our arguments can be made rigorous.

Our strategy is most easily explained for the Brownian step model. The equations of motion are

$$\frac{d}{dt}x_j(t)=a_j(x(t))+\dot b_j(t), \tag{5.1}$$

where $\dot b_j(t)$ is normalized white noise independent for each j. The scaled fluctuation field satisfies then the differential equations

$$\frac{d}{dt}\xi^\varepsilon(f,t)=\sum_j f'(x_j^\varepsilon(t))a_j(x^\varepsilon(t))+\varepsilon\sum_j f''(x_j^\varepsilon(t))$$
$$+\sqrt{\varepsilon}\sum_j f'(x_j^\varepsilon(t))\dot b_j(t). \tag{5.2}$$

Here $x_j^\varepsilon(t)=\varepsilon x_j(\varepsilon^{-1}t)$, and we used the scale invariance of white noise as $\dot b_j(\varepsilon^{-1}t)=\sqrt{\varepsilon}\dot b_j(t)$.

The second term in Eq. (5.2) converges to $\rho\int dx\, f''(x)=0$. The third term converges to a space-time Gaussian measure with covariance

$$\delta(t-t')\rho\int dx\, f'(x)g'(x). \tag{5.3}$$

This uses only that with respect to the distribution given by $|\psi|^2$ we have $\varepsilon\Sigma_j f(\varepsilon x_j)\to\rho\int dx\, f(x)$ in probability as $\varepsilon\to 0$. Equation (5.3) is in accordance with Eq. (4.14), since $c\gamma=\rho$. Thus we are ''only'' left with to show that (recall that $v=0$)

$$\int_0^t ds\sum_j f'(x_j^\varepsilon(s))a_j(x^\varepsilon(s))ds$$
$$\simeq-\int_0^t\sum_j \gamma\sqrt{-\partial^2/\partial x^2}f(x_j^\varepsilon(s))ds. \tag{5.4}$$

If so, the integrated version of Eq. (5.2) becomes a closed equation and agrees with Eq. (4.14).

To establish Eq. (5.4) is certainly the hard part of the matter. In Appendix A we prove that, if we assume the two-point function to scale, the substitution (5.4) holds. This is of interest because the scaling of the two-point function by itself already implies that the fluctuations are Gaussian. I would not know how to obtain such a result otherwise.

A further example for which the substitution (5.4) holds is the Calogero-Sutherland model [18], where the pair potential is proportional to $1/x^2$. If the particles are on a ring $[0,l]$ and if we take all the image potentials into account, then the ground state wave function of the Calogero-Sutherland model is

$$\psi(x_1,\ldots,x_N)=\prod_{i<j=1}^{N}|\sin(\pi(x_i-x_j)/l)|^{\beta/2}, \qquad (5.5)$$

$\beta>0$. By Eq. (4.9) the corresponding drift is given by

$$a_j(x_1,\ldots,x_N)=\frac{\beta}{2}\sum_{i=1,i\neq j}^{N}\frac{\pi}{l}\cot(\pi(x_i-x_j)/l), \qquad (5.6)$$

which in the limit $l\to\infty$ yields

$$a_j(x)=\sum_{i=1,i\neq j}^{N}\frac{\beta}{2}\frac{1}{x_j-x_i}. \qquad (5.7)$$

Compared to Eq. (4.8) only the strength is changed.

The ground state energy per length is $e(\rho)=\frac{1}{24}\beta^2\pi^2\rho^3$ (note that our kinetic energy is $-\frac{1}{2}\Delta$). Therefore, by Eq. (3.26),

$$c=\frac{2}{\pi\beta}, \quad \gamma=\frac{1}{2}\pi\beta\rho. \qquad (5.8)$$

For the Calogero-Sutherland model the substitution in Eq. (5.4) holds without averaging in time. This is somewhat surprising and very particular for the $1/x^2$ potential. To complete the argument one needs the central limit result of Johansson [20],

$$\lim_{\varepsilon\to 0}\left\langle \exp\left[\sum_j f(\varepsilon x_j)\right]\right\rangle=\exp\left[\frac{1}{2}\frac{1}{\pi\beta}\int dk|k||\hat{f}_k|^2\right], \qquad (5.9)$$

for smooth test functions with $\int dx\, f(x)=0$, where $\langle\cdot\rangle$ is the average in the ground state (5.5) in the infinite volume limit. The instructive computation is explained in Appendix B.

Turning to the terrace-step-kink model the situation is more complicated, as may be anticipated from the short range exchange rates [21]. We consider the symmetric case and set $\lambda=0$. Using the method of martingales one can show that the variance of the time-integrated noise is given by

$$\langle c_{01}(\eta)\rangle t=\left(\sum_\eta \psi(\eta)\psi(\eta^{01})(\eta_0-\eta_1)^2\right)t, \qquad (5.10)$$

which is to be compared with the prediction

$$c\gamma=\left.\frac{\partial^2 e}{\partial\lambda^2}\right|_{\lambda=0} \qquad (5.11)$$

from Eq. (4.14). Note that $v=0$, since $\lambda=0$. By second order perturbation theory in λ we obtain

$$\frac{\partial^2 e}{\partial\lambda^2}=\sum_\eta \psi(\eta)\psi(\eta^{01})(\eta_0-\eta_1)^2 -\sum_x\sum_\eta \psi(\eta^{01})(\eta_0-\eta_1)(H-E)^{-1} \times(\eta_x-\eta_{x+1})\psi(\eta^{xx+1}). \qquad (5.12)$$

Thus Eq. (5.11) holds only if the second term vanishes.

For $\Delta=0$ the sum $\Sigma_x(\eta_x-\eta_{x+1})\psi(\eta^{xx+1})$ is the total current J acting on ψ. Since $J\psi=0$, we conclude that for free fermions Eq. (5.11) holds. In this case the two-point function is explicitely known and we can use the argument of Appendix A to prove the conjecture. Alternatively the n-point density correlations can be written in terms of two-point functions. By applying the closed loop theorem, one again concludes that the scaling limit is Gaussian with covariance (3.30) [14,22].

If $\Delta\neq 0$, the second term in Eq. (5.12) is not expected to vanish. This can be verified by expanding $(H-E)^{-1}$ to second order in Δ. Thus the martingale term (5.10) yields the wrong noise strength in the Langevin equation—the hallmark of the so-called nongradient systems. The drift term

$$\int_0^t ds\sum_x f'(\varepsilon x)c_{xx+1}(\eta(\varepsilon^{-1}s)) \qquad (5.13)$$

can no longer be substituted deterministically as in Eq. (5.4). Somewhere hidden there must be an extra noise term. While for short range lattice gases this mechanism is understood by the beautiful work of Varadhan and Yau [21] (cf. also Ref. [23]), the situation looks rather complicated for the long range case considered here.

VI. CONCLUSIONS

We tied together three, at first sight, disconnected pieces: the Luttinger liquid behavior at low energy, surface fluctuations, and the hydrodynamic fluctuation theory for the stochastic dynamics of world lines. In the surface picture one can easily identify the universal low energy limit of the density fluctuations for one-dimensional Fermi fluids. Of course, as an input one needs a variant of the Einstein fluctuation formula. The remainder of the argument is then straightforward. In particular, we show that the parameters of the Tomanaga-Luttinger Hamiltonian must be matched to suitable second derivatives of the energy and length, cf. Eqs. (3.16). To our knowledge, this has not been discussed with sufficient clarity before.

In the interpretation of the world lines as stochastic dynamics the particles have long range interactions, which result from the Dirichlet boundary condition at coinciding positions and thus from the Pauli exclusion principle. We develop a fluctuation theory for the long range case. In fact, the only modification is to replace in the drift term the Laplacian, $-\partial^2/\partial x^2$, by the nonlocal integral operator $\sqrt{-\partial^2/\partial x^2}$. Heuristically, following the arguments of Tomanaga, the $|k|$ decay comes from linearizing the dispersion relation of the particles at the Fermi surface. The not so obvious point is that an interaction changes only the prefactor γ but not the decay law itself. At least for the Calogero-Sutherland Hamiltonian, the mechanism behind such a renormalization could be fully elucidated.

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APPENDIX A

For the Brownian steps we define the structure function $S(k,t)$ by

$$\left\langle\left(\sum_j f(x_j(t))\right)\left(\sum_j g(x_j)\right)\right\rangle=\int dk\,\hat{f}(k)^*g(k)S(k,t), \tag{A1}$$

with $\int dx\, f(x)=0=\int dx\, g(x)$. The average is in the stationary process at infinite volume with density ρ. The existence of this limit and its spatial ergodicity is assumed here. In principle, this could be avoided by considering a finite circle of length l, for which the stationary process exists, and by scaling $l=\varepsilon^{-1}\rho^{-1}$ and $N=\varepsilon^{-1}$. We adopt this strategy for the Calogero-Sutherland model in Appendix B. In essence, Fourier space becomes then discrete, $k\in 2\pi\mathbb{Z}$.

Our real assumption is the scaling of the structure function as

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}S(\varepsilon k,\varepsilon^{-1}t)=\frac{c}{2}|k|e^{-\gamma|k||t|}, \tag{A2}$$

with $c=\sqrt{\rho/e''}$, and $\gamma=\sqrt{\rho e''}$. We want to prove that Eq. (A2) implies

$$\lim_{\varepsilon\to 0}\left\langle\left[\int_0^t ds\sum_j a_j(x^\varepsilon(s))f'(x_j^\varepsilon(s)) + \int_0^t ds\sum_j \gamma\sqrt{-\partial^2/\partial x^2}f(x_j^\varepsilon(s))\right]^2\right\rangle=0. \tag{A3}$$

We note that the time average is needed. This reflects that the system takes a while to adjust to local perturbations.

We work out the square and use that L is symmetric, $\langle F(LG)\rangle=\langle G(LF)\rangle$, and

$$j(f')(x)=\sum_j a_j(x)f'(x_j)+\sum_j \frac{1}{2}f''(x_j)=Ln(f)(x), \tag{A4}$$

$$n(f)(x)=\sum_j f(x_j).$$

Under our scaling the term $n(f'')$ vanishes. We have

$$\begin{aligned}
&\int_0^t ds_1\int_0^t ds_2\langle j^\varepsilon(f',s_1)j^\varepsilon(f',s_2)\rangle\\
&=\int_0^t ds_1\left(\int_0^{s_1}ds_2\langle j^\varepsilon(f')e^{L(s_1-s_2)\varepsilon^{-1}}j^\varepsilon(f')\rangle + \int_{s_1}^t ds_2\langle j^\varepsilon(f')e^{L(s_2-s_1)\varepsilon^{-1}}j^\varepsilon(f')\rangle\right)\\
&=\int_0^t ds_1\left(-\int_0^{s_1}ds_2\frac{\partial}{\partial s_2}\langle j^\varepsilon(f')e^{L(s_1-s_2)\varepsilon^{-1}}n^\varepsilon(f)\rangle + \int_{s_1}^t ds_2\frac{\partial}{\partial s_2}\langle j^\varepsilon(f')e^{L(s_2-s_1)\varepsilon^{-1}}n^\varepsilon(f)\rangle\right)\\
&=-2t\langle j^\varepsilon(f')n^\varepsilon(f)\rangle+2\int_0^t ds\frac{\partial}{\partial s}\langle n^\varepsilon(f)e^{Ls\varepsilon^{-1}}n^\varepsilon(f)\rangle\\
&=-2t\langle j^\varepsilon(f')n^\varepsilon(f)\rangle+2\langle n^\varepsilon(f,t)n^\varepsilon(f)\rangle -2\langle n^\varepsilon(f)n^\varepsilon(f)\rangle.
\end{aligned} \tag{A5}$$

For the first summand we have

$$\langle j^\varepsilon(f')n^\varepsilon(f)\rangle=-\varepsilon^{-1}\frac{1}{2}\sum_j\left\langle\frac{\partial}{\partial x_j}n^\varepsilon(f)\frac{\partial}{\partial x_j}n^\varepsilon(f)\right\rangle. \tag{A6}$$

Therefore, in the limit $\varepsilon\to 0$ we obtain

$$\int dk|\hat{f}|^2(t|k|^2\rho-c|k|(1-e^{-\gamma|k||t|})). \tag{A7}$$

The second term reads, with $g=-\gamma\sqrt{-\partial^2/\partial x^2}f$,

$$\begin{aligned}
&-2\int_0^t ds_1\int_0^t ds_2\langle n^\varepsilon(g,s_1)j^\varepsilon(f',s_2)\rangle\\
&=-2\int_0^t ds_1\left[\int_0^{s_1}ds_2\langle n^\varepsilon(g)e^{L(s_1-s_2)\varepsilon^{-1}}j^\varepsilon(f')\rangle + \int_{s_1}^t ds_2\langle j^\varepsilon(f')e^{L(s_2-s_1)\varepsilon^{-1}}n^\varepsilon(g)\rangle\right]\\
&=-2\int_0^t ds_1\left[-\int_0^{s_1}ds_2\frac{\partial}{\partial s_2}\langle n^\varepsilon(g)e^{L(s_1-s_2)\varepsilon^{-1}}n^\varepsilon(f)\rangle + \int_{s_1}^t ds_2\frac{\partial}{\partial s_2}\langle j^\varepsilon(f')e^{L(s_2-s_1)\varepsilon^{-1}}n^\varepsilon(g)\rangle\right]\\
&=-2\int_0^t ds[-2\langle n^\varepsilon(g)n^\varepsilon(f)\rangle+2\langle n^\varepsilon(g)e^{Ls\varepsilon^{-1}}n^\varepsilon(f)\rangle].
\end{aligned} \tag{A8}$$

By assumption, this expression converges to

$$\int dk|\hat{f}|^2(-2t\gamma ck^2+2|k|c(1-e^{-\gamma|k|t})). \tag{A9}$$

The third term is given by

$$\int_0^t ds_1\int_0^t ds_2\langle n^\varepsilon(g,s_1)n^\varepsilon(g,s_2)\rangle, \tag{A10}$$

which by assumption converges to

$$\int_0^t ds_1\int_0^t ds_2\int dk|\hat{f}|^2\gamma^2\frac{c}{2}|k|^3e^{-\gamma|k||s_1-s_2|}$$

$$=\int dk|\hat{f}|^2(t\gamma ck^2-c|k|(1-e^{-\gamma|k|t})). \tag{A11}$$

Using the fact that $\gamma c=\rho$, the sum of the three terms vanishes, as claimed.

To complete the argument, one notes that

$$M^\varepsilon(f,t)=\int_0^t ds\sum_j f(\varepsilon x_j(\varepsilon^{-1}s))db_j(s) \tag{A12}$$

is a Martingale function with square

$$M(f,t)^2=\int_0^t ds\,\varepsilon\sum_j f'(\varepsilon x_j(\varepsilon^{-1}s))^2+M_1^\varepsilon(f,t), \tag{A13}$$

where $M_1^\varepsilon(f,t)$ is again a Martingale function. We now assume the validity of a law of large numbers as

$$\lim_{\varepsilon\to 0}\varepsilon\sum_j g(\varepsilon x_j)=\rho\int dx\,g(x). \tag{A14}$$

Then, under suitable tightness, in the limit $\varepsilon\to 0$

$$M(f,t)^2=t\int dx\,f'(x)^2+M_1(f,t), \tag{A15}$$

with both $M(f,t)$ and $M_1(f,t)$ Martingale functions. This implies that $M(f,t)$ is Brownian motion in t with covariance $\int dx\,f'(x)^2$. We refer to Ref. [18] for a more complete discussion.

APPENDIX B

We consider the Calogero-Sutherland model on a finite ring of length l. The density is $\rho=N/l$. We scale the positions with ε^{-1}. Setting $\varepsilon=2\pi/l$, we scale back to the circle $[0,2\pi]$. Then $N=\varepsilon^{-1}2\pi\rho$. Since the density scales, we set $2\pi\rho=1$ and use N instead of ε. After these transformations the ground state $\psi^2=e^{-\beta V}$ with the logarithmic potential

$$V=-\sum_{i<j=1}^{N}\ln|\sin((x_i-x_j)/2)|. \tag{B1}$$

The drift is then

$$a_j(x)=-\frac{\partial}{\partial x_j}\beta V(x)=\frac{\beta}{2}\sum_{\substack{i=1\\ i\neq j}}^{N}\cot((x_j-x_i)/2). \tag{B2}$$

For this model we show the validity of substitution (5.4) without averaging in t.

More explicitly, we have to show that

$$\frac{1}{N}\sum_{j=1}^{N}a_j(x)f'(x_j)=\frac{1}{2N}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\beta\cot((x_j-x_i)/2)f'(x_j)$$

$$\simeq\sum_{j=1}^{N}g(x_j), \tag{B3}$$

where

$$g(x)=\frac{\beta}{4\pi}\int_0^{2\pi}dy\cot((x-y)/2)f'(y), \tag{B4}$$

in the sense of the Cauchy principal part. Since we are on a ring, f has discrete Fourier coefficients $\hat{f}_k=(1/2\pi)\int_0^{2\pi}dx\,e^{ikx}f(x)$. We assume that $\Sigma_{|k|}|k|^3\hat{f}_k<\infty$. Note that $\hat{g}_k=-(\beta/2)|k|\hat{f}_k$. To show Eq. (B3) we use a result of Johansson [20] which states

$$\lim_{N\to\infty}\frac{1}{N}\left\langle\sum_{j=1}^{N}h(x_j)\right\rangle=\hat{h}_0, \tag{B5}$$

and, provided $\hat{h}_0=0=\hat{f}_0$,

$$\lim_{N\to\infty}\left\langle\left(\sum_{j=1}^{N}h(x_j)\right)\left(\sum_{i=1}^{N}f(x_i)\right)\right\rangle=\frac{2}{\beta}\sum_k|k|\hat{h}_k^*\hat{f}_k. \tag{B6}$$

Let us work out the square. The first term is

$$\left\langle\left(\sum_{j=1}^{N}g(x_j)\right)^2\right\rangle, \tag{B7}$$

which by Eq. (B6) tends in the limit $N\to\infty$ to

$$\frac{2}{\beta}\sum_k|k||\hat{g}_k|^2=\frac{\beta}{2}\sum_k|k|^3|\hat{f}_k|^2. \tag{B8}$$

For the second term we use $(\partial/\partial x_i)e^{-\beta V}=a_ie^{-\beta V}$. Then

$$-2\frac{1}{N}\left\langle\left(\sum_{j=1}^{N}a_jf'(x_j)\right)\left(\sum_{i=1}^{N}g(x_i)\right)\right\rangle$$

$$=\frac{2}{N}\sum_{i,j=1}^{N}\left\langle\frac{\partial}{\partial x_j}(f'(x_j)g(x_i))\right\rangle$$

$$=\frac{2}{N}\left\langle\left(\sum_{j=1}^{N}f''(x_j)\right)\left(\sum_{i=1}^{N}g(x_i)\right)\right\rangle$$

$$+\frac{2}{N}\left\langle\sum_{j=1}^{N}f'(x_j)g'(x_j)\right\rangle. \tag{B9}$$

By Eq. (B6) the first summand vanishes, and by Eq. (B5) the second summand converges to

$$2\sum_k |k|^2 \hat{f}_k^* \hat{g}_k = -2\beta \sum_k |k|^3 |\hat{f}_k|^2. \tag{B10}$$

For term number 3 we use $(\partial^2/\partial x_i \partial x_j)e^{-\beta V} = a_i a_j e^{-\beta V} + (\partial a_i/\partial x_j)e^{-\beta V}$. Then

$$\frac{1}{N^2}\left\langle \left(\sum_{j=1}^N a_j f'(x_j)\right)\left(\sum_{i=1}^N a_i f'(x_i)\right)\right\rangle \tag{B11}$$

$$= \frac{1}{N^2}\sum_{i,j=1}^N \left\langle \frac{\partial}{\partial x_i}\frac{\partial}{\partial_j}(f'(x_i)f'(x_j)) - \left(\frac{\partial}{\partial x_j} a_i\right) f'(x_i) f'(x_j)\right\rangle$$

$$= \frac{1}{N^2}\left(\left\langle \left(\sum_{j=1}^N f''(x_j)\right)\left(\sum_{i=1}^N f''(x_i)\right)\right\rangle + \left\langle \sum_{j=1}^N f''(x_j)^2 \right\rangle + 2\left\langle \sum_{j=1}^N f'(x_j) f'''(x_j)\right\rangle\right)$$

$$+ \frac{1}{N^2}\frac{\beta}{8}\left\langle \sum_{i,j=1}^N |\sin((x_i - x_j)/2)|^{-2} \times [f'(x_i) - f'(x_j)]^2 \right\rangle.$$

By Eq. (B5) the first summand vanishes. The second summand converges to

$$\frac{\beta}{8}(2\pi)^{-2}\int_0^{2\pi} dx \int_0^{2\pi} dy |\sin((x-y)/2)|^{-2}(f'(x) - f'(y))^2$$

$$= \frac{\beta}{2}\sum_k |k|^3 |\hat{f}_k|^2. \tag{B12}$$

Adding the three terms we conclude that the sum vanishes.

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